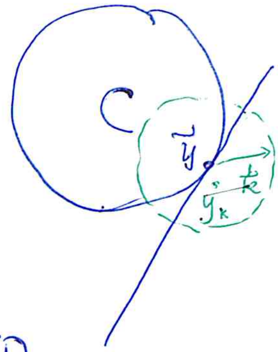
Thm 1.4 C closed & convex, $\vec{y}$ boundary pt of C

Then $\exists$ a supporting hyperplane at $\vec{y}$



pf: $\vec{y}$ bddary pt

$\therefore \forall k, \exists \vec{y}_k \in B_{\frac{1}{k}}(\vec{y})$ s.t. $\vec{y}_k \notin C$

Clearly $\lim_{k\to\infty} \vec{y}_k = \vec{y}$ ①

for each $\vec{y}_k$, $\exists X_k \equiv \{ \vec{x} \mid \vec{a}_k^T\vec{x} = \vec{a}_k^T\vec{y}_k \} \ni \vec{y}_k$

s.t. $C \subseteq X_k^+ = \{ \vec{x} \mid \vec{a}_k^T x > \vec{a}_k^T\vec{y}_k \}$ ②

w.l.o.g. $|a_k| = 1$, $\forall k$.

Since $\{a_k\}$ is bdd, $\exists$ convergence subsequence

$$\lim_{j\to\infty} \vec{a}_{k_j} = \vec{a}$$ ③

Define $X \equiv \{ \vec{x} \mid \vec{a}^T\vec{x} = \vec{a}^T y \}$ is a supporting hyperplane.

(i) $\vec{y} \in X$ clearly.

(ii) $C \subseteq X^+ \equiv \{ \vec{x} \mid \vec{a}^T\vec{x} \geq a^T y \}$? :

$$\forall \vec{x} \in C, \quad \vec{a}^T\vec{x} \overset{③}{=} \lim_{j\to\infty} \vec{a}_{k_j}\vec{x} \overset{②}{\geq} \lim_{j\to\infty} \vec{a}_{k_j}\vec{y}_{k_j} \overset{①}{=} \vec{a}^T\vec{y} \ \#$$

Thm 1.8 C closed convex & bdd from below.

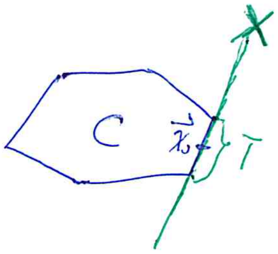
Then every supporting hyperplane of C has an extreme pt of C

Pf: Let $\vec{x}_0 \in C$ be a boundary pt

& $X = \{\vec{x} \mid \vec{a}^T x = z\} \ni \vec{x}_0$ is a supporting hyperplane at $\vec{x}_0$

w.l.o.g. $C \subseteq X^+ = \{\vec{x} \mid \vec{a}^T x \geq z\}$

Define $T = X \cap C$



N.T.P. (i) $T \neq \phi$, (ii) ex. pt of $T \Rightarrow$ ex pt of C

(iii) $\exists$ ex pt of T.

(i) $T \neq \phi$: $\vec{x}_0 \in C$, $\vec{x}_0 \in X \Rightarrow \vec{x}_0 \in T$

(ii) "ex pt of $T \Rightarrow$ ex pt of C" $\Leftrightarrow$ "not ex pt of $C \Rightarrow$ not ex pt of T"

Let $\vec{t}$ not ex. pt of C

Case 1: $\vec{t} \notin T \Rightarrow \vec{t}$ cannot be ex pt of T

Case 2: $\vec{t} \in T. \Rightarrow \vec{t} \in X \Rightarrow \vec{a}^T \vec{t} = z$

since $\vec{t}$ not ex pt of $C \Rightarrow \exists\ \vec{x}_1, \vec{x}_2 \in C$ s.t.

$$\vec{t} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \qquad \lambda \in (0,1)$$

$$\Rightarrow \underbrace{\vec{a}^T \vec{t}}_{= z} = \lambda \underbrace{\vec{a}^T \vec{x}_1}_{\geq z} + (1-\lambda) \underbrace{\vec{a}^T \vec{x}_2}_{\geq z}$$

$\Rightarrow \vec{a}^T x_1 = z$ & $\vec{a}^T \vec{x}_2 = z$

$\Rightarrow \vec{x}_1, \vec{x}_2 \in X$

$\Rightarrow \vec{x}_1, \vec{x}_2 \in T = X \cap C$

$\therefore$ have found $\vec{x}_1, \vec{x}_2 \in T$ s.t.

$$\vec{t} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2$$

$\Rightarrow \vec{t}$ is not an ex. pt of T.

(iii) $\exists$ ex pt of T. ?

$\forall \vec{t} \in T$ write them as $\vec{t} = (t_1, .., t_n)$ $\quad T = \{ \vec{t} \mid (t_1, .., t_n) \}$

Let $t^1 = \min_{\forall \vec{t} \in T} t_1$ $\quad$ (T is closed as C is bdd below)

if $\exists$ only 1 $\vec{t}$ that attain the minimum, denote it as $\vec{t}^1$

If not $t^2 = \min_{\substack{\forall t \in T, \\ s.t.\ t_1 = t^1}} t_2$

if $\exists$ only 1 $\vec{t}$ that attain the minimum, denote it by $\vec{t}^2$

Process must stop, say at j.

Claim $\vec{t}^j$ is an extreme pt of T.

Suppose not, i.e. $\exists$ $\vec{y}_1, \vec{y}_2 \in T$ s.t.

$$\vec{t}^j = \lambda \vec{y}_1 + (1-\lambda) \vec{y}_2 \qquad \lambda \in (0,1)$$

Consider the jth coordinate

$$t^j = \lambda \underset{t^j}{\underset{\vee}{(y_{1j})}} + (1-\lambda) \underset{t^j}{\underset{\vee}{(y_{2j})}} \quad (\because \text{uniqueness})$$

But this is impossible $\#$